Q.2 a. For an energy signal $x(t)$ with energy E_x, determine the energy of the following signals:

(i) $x(t-T)$ (ii) $x(at)$

(iii) $x(at-b)$ (iv) $ax(t)$

Answer:

Ans. (i) By definition the energy contained in the signal $x(t-T)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t-T)|^2 dt = \int_{-\infty}^{\infty} |x(\tau)|^2 d\tau = E_x$$

(ii) By definition the energy contained in the signal $x(at)$ is given by

$$E = \int_{-\infty}^{\infty} |x(at)|^2 dt = \frac{1}{a}\int_{-\infty}^{\infty} |x(\tau)|^2 d\tau = \frac{E_x}{a}$$

(iii) By definition the energy contained in the signal $x(at-b)$ is given by

$$E = \int_{-\infty}^{\infty} |x(at-b)|^2 dt = \frac{1}{a}\int_{-\infty}^{\infty} |x(\tau)|^2 d\tau = \frac{E_x}{a}$$

(iv) By definition the energy contained in the signal $ax(t)$ is given by

$$E = \int_{-\infty}^{\infty} |ax(t)|^2 dt = a^2\int_{-\infty}^{\infty} |x(t)|^2 dt = a^2 E_x$$

b. If $x(t) * h(t) = y(t)$, then show that $x(at) * h(at) = \frac{1}{|a|} y(at)$

Answer:

Ans. Case-I: $a > 0$. By definition

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = y(t)$$

$$x(at) * h(at) = \int_{-\infty}^{\infty} x(a\tau)h(a(t-\tau))d\tau = \int_{-\infty}^{\infty} x(a\tau)h(at-a\tau)d\tau$$

A change of variables is performed by letting $a\tau = \alpha$, which also yields $d\tau = \frac{1}{a}d\alpha$, $\alpha \to \infty$ as $\tau \to \infty$, and $\alpha \to -\infty$ as $\tau \to -\infty$. Therefore,

$$x(at) * h(at) = \frac{1}{a}\int_{-\infty}^{\infty} x(\alpha)h(at-\alpha)d\alpha = \frac{1}{a}y(at)$$

Case II: $a < 0$. By definition

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = y(t)$$

$$x(-at) * h(-at) = \int_{-\infty}^{\infty} x(-a\tau)h(-a(t-\tau))d\tau = \int_{-\infty}^{\infty} x(-a\tau)h(-at+a\tau)d\tau$$

A change of variables is performed by letting $-a\tau = \alpha$, which also yields $d\tau = -\frac{1}{a}d\alpha$, $\alpha \to \infty$ as $\tau \to -\infty$, and $\alpha \to -\infty$ as $\tau \to \infty$. Therefore,

$$x(at) * h(at) = \frac{1}{a}\int_{-\infty}^{\infty} x(\alpha)h(-at-\alpha)d\alpha = \frac{1}{a}y(-at)$$

From the above two cases it is evident that

$$x(at) * h(at) = \frac{1}{|a|}y(at)$$

Q.3 a. Let $X[k]$ represent the DTFS coefficients of the periodic sequence $x(n)$ with period N. Find the DTFS coefficients of $(-1)^n x(n)$

Answer:

Ans. Consider the signal

$$(-1)^n x(n) = e^{j\pi n} x(n) = e^{j\frac{2\pi}{N}\frac{N}{2}n} x(n)$$

We know that if

$$x(n) \longleftrightarrow X_k$$

then using frequency-shifting property, we obtain

$$e^{j\frac{2\pi}{N}\frac{N}{2}n} x(n) \longleftrightarrow X_{k-\frac{N}{2}}$$

b. Suppose we are given the following information about a periodic signal $x(n)$ with period $N = 8$ and Fourier series coefficients $X[k]$:

(i) $X[k] = -X[k-4]$ (ii) $x(2n+1) = (-1)^n$

Sketch one period of $x(n)$

Answer:

Ans. Consider the signal

$$(-1)^n x(n) = e^{j\pi n} x(n) = e^{j\frac{2\pi}{N}\frac{N}{2}n} x(n)$$

We know that if

$$x(n) \longleftrightarrow X_k$$

then using frequency-shifting property, we obtain

$$e^{j\frac{2\pi}{N}\frac{N}{2}n} x(n) \longleftrightarrow X_{k-\frac{N}{2}}$$

In this case $N = 8$, therefore

$$(-1)^n x(n) \longleftrightarrow X_{k-4}$$

Since, it is given that $X_k = -X_{k-4}$, we have

$$\begin{aligned} (-1)^n x(n) &\longleftrightarrow -X_k \\ (-1)^n x(n) &= -x(n) \\ x(n)[1 + (-1)^n] &= 0 \end{aligned}$$

This implies that

$$x(n) = 0 \quad \text{for} \quad n = 0, \pm 2, \pm 4, \pm 6, \cdots$$

From the given information $x(2n+1) = (-1)^n$, we get

$$x(1) = 1, \quad x(3) = -1, \quad x(5) = 1, \quad x(7) = -1$$

Therefore, over one period $0 \leq n \leq 7$, $x(n)$ is defined as

$$x(n) = \begin{cases} 0 & n = 0, 2, 4, 6 \\ 1 & n = 1, 5 \\ -1 & n = 3, 7 \end{cases}$$

Q.4 a. Given that $x(t)$ has the Fourier transform $X(\omega)$, express the Fourier transforms of the signal listed below in terms of $X(\omega)$.

(i) $x_1(t) = x(1-t) + x(-1-t)$ (ii) $x_2(t) = x(3t-6)$

Answer:

Ans. Given that

$$x(t) \longleftrightarrow X(\omega)$$

(a) Using the time shifting property, we have

$$x(t+1) \longleftrightarrow X(\omega)e^{j\omega}$$
$$\text{and} \quad x(t-1) \longleftrightarrow X(\omega)e^{-j\omega}$$

Now, using the time reversal property, we have

$$x(-t+1) \longleftrightarrow X(-\omega)e^{-j\omega}$$
$$\text{and} \quad x(-t-1) \longleftrightarrow X(-\omega)e^{j\omega}$$

Therefore,

$$\begin{aligned} x_1(t) &\longleftrightarrow X_1(\omega) \\ x(1-t) + x(-1-t) &\longleftrightarrow X(-\omega)e^{-j\omega} + X(-\omega)e^{j\omega} \\ x(1-t) + x(-1-t) &\longleftrightarrow 2X(-\omega)\cos\omega \\ \mathcal{F}[x_1(t)] = X_1(\omega) &= 2X(-\omega)\cos\omega \end{aligned}$$

b. Find the Fourier transform $G(\omega)$ of the signal $g(t) = \frac{1}{\pi t}$

Answer:

Ans. Define $X(\omega) = \frac{1}{\pi\omega}$ by replacing t with ω in the expression of $g(t)$. We know that

$$\begin{aligned} \text{sgn}(t) &\longleftrightarrow \frac{2}{j\omega} \\ \frac{j}{2\pi}\text{sgn}(t) &\longleftrightarrow \frac{1}{\pi\omega} \\ x(t) &\longleftrightarrow X(\omega) \end{aligned}$$

$$x(t) = \frac{j}{2\pi}\text{sgn}(t) \quad \text{and} \quad X(\omega) = \frac{1}{\pi\omega}$$

$$x(-\omega) = \frac{j}{2\pi}\text{sgn}(-\omega) = -\frac{j}{2\pi}\text{sgn}(\omega) \quad \text{and} \quad X(t) = \frac{1}{\pi t}$$

The duality property of the fourier transform states that if

$$x(t) \longleftrightarrow X(\omega)$$

then,

$$\begin{aligned} X(t) &\longleftrightarrow 2\pi x(-\omega) \\ \frac{1}{\pi t} &\longleftrightarrow -2\pi\frac{j}{2\pi}\text{sgn}(\omega) \\ \frac{1}{\pi t} &\longleftrightarrow -j\,\text{sgn}(\omega) \end{aligned}$$

Therefore,

$$\mathcal{F}\left[\frac{1}{\pi t}\right] = -j\,\text{sgn}(\omega) = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases}$$

Q.5 a. Given that $x(n)$ has the Fourier transform $X(e^{j\omega})$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$.

(i) $x_1(n) = (n-1)^2 x(n)$ (ii) $x_2(n) = e^{jn\pi/2} x(n+2)$

Answer:

Ans. (i) Consider the given signal $x_1(n) = (n-1)^2 x(n) = n^2 x(n) - 2nx(n) + x(n)$.
Using the differentiation in frequency domain property, we get

$$nx(n) \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

Using the differentiation in frequency domain property again, we have

$$n[nx(n)] \longleftrightarrow j\frac{d}{d\omega}\left(j\frac{dX(e^{j\omega})}{d\omega}\right)$$
$$n^2 x(n) \longleftrightarrow -\frac{d^2X(e^{j\omega})}{d\omega}$$

Therefore,

$$\mathcal{F}[x_1(n)] = \mathcal{F}[(n-1)^2 x(n)]$$
$$= \mathcal{F}[n^2 x(n) - 2nx(n) + x(n)]$$

$$X_3(e^{j\omega}) = -\frac{d^2X(e^{j\omega})}{d\omega} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$
$$X_3(e^{j\omega}) = -\frac{d^2X(e^{j\omega})}{d\omega} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

(ii) Using the time shifting property, we get

$$x(n+2) \longleftrightarrow X(e^{j\omega})e^{j2\omega}$$

Using the frequency shifting property on this, we get

$$x_2(n) = e^{j\frac{\pi}{2}n}x(n+2) \longleftrightarrow X(e^{j(\omega-\frac{\pi}{2})})e^{j2(\omega-\frac{\pi}{2})}$$
$$X_2(e^{j\omega}) = X(e^{j(\omega-\frac{\pi}{2})})e^{j2(\omega-\frac{\pi}{2})}$$

b. Let the sequence $x(n)$ be a real sequence and let $X(e^{j\omega}) = \text{DTFT}[x(n)]$

(i) Prove that the magnitude spectrum is an even function, that is, $|X(e^{j\omega})| = |X(e^{-j\omega})|$

(ii) Prove that the phase spectrum is an odd function, that is, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$

Answer:

Ans. Given that

$$x(n) \longleftrightarrow X(e^{j\omega})$$

In general the Fourier transform $X(e^{j\omega})$ is a complex function of the real variable ω and can be written in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}),$$

where $X_R(e^{j\omega})$ and $X_I(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$.
(a) The magnitude spectrum $|X(e^{j\omega})|$ is given by

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$
$$|X(e^{-j\omega})| = \sqrt{X_R^2(e^{-j\omega}) + X_I^2(e^{-j\omega})}$$

Since $X_R(e^{j\omega}) = X_R(e^{-j\omega})$, and $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$, we obtain

$$|X(e^{-j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$
$$|X(e^{-j\omega})| = |X(e^{j\omega})|$$

Since $X_R(e^{j\omega}) = X_R(e^{-j\omega})$, and $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$, we obtain

$$|X(e^{-j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$
$$|X(e^{-j\omega})| = |X(e^{j\omega})|$$

The magnitude spectrum $|X(e^{j\omega})|$ is an even function of ω.

(b) We know that the Fourier transform $X(e^{j\omega})$ is a complex function of the real variable ω and can be written in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$
$$\theta(\omega) = \angle X(e^{j\omega}) = \tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$
$$\angle X(e^{-j\omega}) = \tan^{-1}\left[\frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})}\right]$$

Since $X_R(e^{j\omega}) = X_R(e^{-j\omega})$, and $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$, we obtain

$$\angle X(e^{-j\omega}) = \tan^{-1}\left[-\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$
$$\angle X(e^{-j\omega}) = -\tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$
$$\angle X(e^{-j\omega}) = -\angle X(e^{j\omega})$$

The above equation implies that the phase spectrum $\angle X(e^{j\omega})$ is an odd function of ω.

Q.6 a. A waveform $x(t) = 10 + 10\sin(500t)$ is to be sampled periodically and reproduced from these samples. Find the maximum allowable time interval between sample values. How many sample values are required to be stored in order to produce 2 seconds of this waveform?

Answer:

Ans. Given that

$$x(t) = 10 + 10\sin(500t)$$

The maximum frequency present in the given signal $x(t)$is

$$f_{max} = \frac{500}{2\pi} = 79.58 \text{ Hz}$$

Therefore the Nyquist rate (minimum sampling frequency) is given by

$$f_s = 2f_{max} = 159.16 \text{ Hz}$$

Thus, the maximum allowable time interval between the sample values is given by

$$T_s = \frac{1}{f_s} = \frac{1}{159.16} = 6.28 \text{ m sec.}$$

The number of sample values required to be stored in order to produce 1 second of this waveform is given by

$$\text{Number of samples} = \frac{2\,\text{sec.}}{6.28\,\text{m sec.}} = 318.5$$

b. A signal $x(t) = \sin(\pi t)/(\pi t)$ is sampled by $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n/2)$. Determine and sketch the sampled signal and its Fourier transform.

Answer:

Ans. Let $x_s(t)$ be the sampled signal. Therefore,

$$x_s(t) = x(t)s(t) = \frac{\sin(\pi t)}{\pi t} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{2}\right) \sum_{n=-\infty}^{\infty} \frac{\sin\left(\pi \frac{n}{2}\right)}{\pi \frac{n}{2}} \delta\left(t - \frac{n}{2}\right)$$

Q.7 a. Show that for an LTI system, when the input is $x(t) = e^{s_o t}u(t)$, the output is of the form $y(t) = H(s_o)e^{s_o t}u(t)$. How is $H(s_o)$ related to the impulse response of the system?

Answer:

Ans. We know that the input and output of an LTI system is related by

$$\begin{aligned} y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau \\ &= e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau \\ y(t) &= H(s_0)e^{s_0 t} \end{aligned}$$

where

$$H(s_0) = \int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau = \mathcal{L}[h(t)]\Big|_{s=s_0}$$

b. Determine the impulse response $h(t)$ of a system having a double-order pole at $s = -a$ and a zero at $s = -b$, where $a, b > 0$ and $b - a = B$. It is also given that $h(0) = 2$

Answer:

Ans. Since $h(t)$ of the system having a double-order pole at $s = -a$, and a zero at $s = -b$, we may assume that $H(s)$ is of the form

$$X(s) = \frac{K(s+b)}{(s+a)^2}$$

$$X(s) = \frac{Ks}{(s+a)^2} + \frac{Kb}{(s+a)^2}$$

$$X(s) = KsG(s) + KbG(s)$$

where

$$G(s) = \frac{1}{(s+a)^2}$$

Taking its inverse Laplace transform, we get

$$g(t) = te^{-at}u(t)$$

Also, because

$$X(s) = KsG(s) + KbG(s)$$

Q.8 a. Apply the final-value theorem of z-transform to determine $x(\infty)$ for the signal $x(n)=\begin{cases}1, & \text{if } n \text{ is even}\\ 0, & \text{otherwise}\end{cases}$

Answer:

Ans. Given that

$$x(n)=\begin{cases}1 & \text{if } n \text{ is even}\\ 0 & \text{otherwise}\end{cases}$$

From the definition of the unilateral z-transform, we have

$$X(z)=\sum_{n=0}^{\infty} x(n)z^{-n}=\sum_{\substack{n=0\\ n\ \text{even}}}^{\infty}(1)z^{-n}$$

Substituting $n=2r$, where r is varying from 0 to ∞, we obtain

$$X(z)=\sum_{r=0}^{\infty} z^{-2r}=\sum_{r=0}^{\infty}\left(z^{-2}\right)^r=\frac{1}{1-z^{-2}}, \qquad |z^{-2}|<1\longrightarrow |z|>1$$

From the final value theorem, we have

$$x(\infty)=\lim_{z\to 1}(1-z^{-1})X(z)=\lim_{z\to 1}\frac{1-z^{-1}}{1-z^{-2}}$$

$$=\lim_{z\to 1}\frac{1-z^{-1}}{(1-z^{-1})(1+z^{-1})}=\lim_{z\to 1}\frac{1}{1+z^{-1}}=\frac{1}{2}$$

b. An LTI system is characterized by the system function

$$H(z)=\frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

Specify the ROC of $H(z)$ and determine the impulse response $h(n)$ for the following conditions:

(i) The system is causal and unstable
(ii) The system is noncausal and stable
(iii) The system is anticausal and unstable

Answer:

Ans. Given that

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$
$$= \frac{3z^2 - 4z}{z^2 - 3.5z + 1.5}$$
$$\frac{H(z)}{z} = \frac{3z - 4}{z^2 - 3.5z + 1.5}$$
$$= \frac{3z - 4}{(z - 0.5)(z - 3)}$$

Using partial-fraction expansion, we obtain

$$\frac{H(z)}{z} = \frac{1}{z - 0.5} + \frac{2}{z - 3}$$
$$H(z) = \frac{z}{z - 0.5} + \frac{2z}{z - 3}$$
$$= \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

This system has poles at $z = 0.5$ and $z = 3$.
(i) For this system to be causal and unstable, the ROC of $H(z)$ is the region in the z-plane outside the outermost pole and it must not include the unit circle. Therefore, the ROC is the region, $|z| > 3$.
Since the ROC, $|z| > 3$, is the region in the z-plane outside the outermost pole, all the poles correspond to causal (right-sided) signals. Now, consider

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}, \qquad |z| > 3$$

The inverse z-transform yields

$$h(n) = (0.5)^n u(n) + 2(3)^n u(n)$$

(ii) For this system to be noncausal and stable, the ROC of $H(z)$ is a ring in the z-plane and it must include the unit circle. Therefore, the ROC is the region, $0.5 < |z| < 3$.
The pole of the first term is at 0.5. The ROC has a radius greater than the pole at $z = 0.5$, so this pole corresponds to causal (right-sided) signal. Therefore,

$$(0.5)^n u(n) \longleftrightarrow \frac{1}{1 - 0.5z^{-1}}$$

The second term has a pole at $z = 3$. The ROC has a radius less than the pole at $z = 3$, so this pole corresponds to the anti-causal (left-sided) signal. Therefore,

$$-2(3)^n u(-n-1) \longleftrightarrow \frac{2}{1 - 3z^{-1}}$$

and hence, we obtain

$$h(n) = (0.5)^n u(n) - 2(3)^n u(-n-1)$$

(iii) For this system to be anti-causal and unstable, the ROC of $H(z)$ is the region in the z-plane inside the innermost pole and it must not include the unit circle. Therefore, the ROC is the region, $|z| < 0.5$.
Since the ROC, $|z| < 0.5$, is the region in the z-plane inside the innermost pole, all the poles correspond to anti-causal (right-sided) signals. Now, consider

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}, \qquad |z| < 0.5$$